Literal Singular-Value-Based Flight Control System Design Techniques

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Recent control system research has resulted in new methods for assessing stability and performance robustness with respect to plant and controller uncertainty. The emphasis has been on using frequency domain matrix singular values for multi-input/multi-output systems. Problems originating from aircraft flight control systems have been an important motivation for these developments, and applications are currently being carried out at government research centers and in industry. These new robustness methods are, however, only slowly becoming part of the working flight control designer's practical tools. Before becoming routine, means must be provided for better physical insight. This paper presents a distinctly different interpretation based on literal (symbolic) formulations of singular values. This approach is an outgrowth of and is related to classical multivariable control systems methods based on coupling numerators. A lateral-directional flight control system design for an advanced fighter is used as an example. Robustness of stability with respect to input uncertainties is first assessed using standard numerical techniques. Literal approximations are then given for open- and closed-loop singular values. The literal approximations are consistent with the numerical results. More importantly, they provide a means for decomposing and diagnosing robustness problems by providing explicit connections to critical aircraft and controller parameters. Both unstructured and structured uncertainties are treated, using respectively literal approximations of regular and structured singular values.

Introduction

ROBUSTNESS issues for single-input/single-output systems have long been understood and appreciated by designers. The objective is to design a controller which is, in some sense, tolerant and forgiving. Extending these results to multivariable systems has been the focus of a stream of research in recent years. A hallmark of the new approaches has been a renewed emphasis on frequency domain techniques, after a period in which these methods had often been viewed as mature or even passe.

The key to extending the frequency domain robustness methods has been the generalization of gain using singular values of a matrix.[1,2] Reformulations of this metric, notably using structured singular values,[3,4] continue to be central to most of the new methods. Most of the work has been done in a general context using abstract $\dot{x} = Ax + Bu$ type linear systems. A byproduct of working at this level of abstraction is that the tools of the trade are largely computer programs implementing very general and sophisticated numerical methods. This focus on the computational aspects of the problem is a characteristic that multivariable frequency domain robustness methods have in common with earlier time domain and optimal control methods.

Aircraft flight control system (FCS) design has been an important motivation for new robustness methods. Flight control related work has been done in industry, at NASA,[5,6] the Air Force,[7] and at universities.[8] Aircraft manufacturers also have experimented with these methods but, in general, they are a long way from routine working tools. This is to be expected in the hard-nosed environment of specific aircraft projects, where new theoretical tools have to compete against techniques that have stood the test of time on many successful projects.

An early impediment to the use of singular-value-based robustness tests was the lack of widely available mature software. Commercial packages are now available at a reasonable cost, so this is less of an issue. However there are more subtle factors at work, generally related to incorporating the thinking process and expertise of the flight control designer. General theory is not of any great practical interest to the designer faced with a myriad of quantitative and qualitative requirements.

Of more concern is to expose and bound uncertainties in stability derivatives, actuator and sensor dynamics, and other aircraft parameters. To do this, the designer typically studies single variations using a variety of conventional measures such as gain, phase, and time-delay margins, peak amplification ratios, dominant mode characteristics, and so on.

Coupling several variations to assess overall robustness is where expertise and experience are most useful. A designer's knowledge of the particulars of a given airframe is the secret weapon that makes this possible. Multivariable singular-value-based robustness tests present an opportunity to do this traditional job more efficiently and rigorously, but as currently applied these tests are strictly numerical, with the result that the designer loses the physical insight needed to diagnose a robustness problem.

Flight control designs in the future will have larger uncertainties in aircraft operating over greatly expanded flight envelopes. Control systems will be reconfigurable, perhaps even in real-time in the presence of failures or damage. New robustness procedures offer a solid approach to these problems. The need has never been greater for tying together the existing academic generality with the physical insight of a good designer. This paper attempts to improve this relationship; specifically, we will do the following.

1) Connect multivariable frequency domain singular value methods to classical coupling numerator based multiloop methods.[9]

2) Develop physical insights between the singular values and aircraft and controller dynamics by using literal approximate factors.

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3) Provide a recipe for the identification of the most important aircraft and controller parameter uncertainties.

Example Problem

Throughout the paper, a lateral-directional FCS design for an advanced fighter aircraft is used as an example.[10] A state space model for the aircraft is shown in Table 1 ($M = 0.6$, 35,000 ft, 50% fuel). The low-frequency effects of the rudder and roll control actuators have been modeled as first-order lags (with poles at 20 and 25 rad/s, respectively). The unaugmented aircraft characteristics were contrived to exhibit several generic problems characteristic of lateral control at moderately high angles of attack including: 1) high roll subsidence time constant ($T_R = 2.34$ s, MIL-F-8785C level 3); 2) low Dutch roll damping ratio ($\zeta_d = 0.0208$, MIL-F-8785C level 3); and 3) undesirable roll-due-to-aileron control numerator ($\omega_\phi/\omega_d > 1$, with low damping for the complex zero[9]).

A simple FCS design has been formulated that satisfactorily solves the preceding problems and, in addition, provides bank angle control. As shown in Fig. 1, this system consists of a washed-out stability axis yaw damper ($r_s \to \delta_r$), a body axis roll damper ($p_b \to \delta_a$), and an autopilot bank angle command and hold loop ($\phi \to \delta_a$). In traditional designs, the feedbacks would be the output of three sensors. The inner loop stability augmentation feedbacks would be provided by yaw- and roll-rate gyros, with r_s being a blended combination of r_b and p_b. The outer loop autopilot ϕ would be provided by a distinct vertical gyro. Next-generation fighters and other aircraft can be expected to use integrated inertial reference assemblies in which one set of (redundant) gyros provides both attitude and attitude rate information through strapdown calculations.

Table 1 Lateral-directional airframe model

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x = \begin{pmatrix} \beta \\ p_b \\ r_b \\ \phi \end{pmatrix} \qquad u = \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix} \qquad y = \begin{pmatrix} p_b \\ \phi \\ r_s \end{pmatrix}$$

where

β = side slip, rad
p_b = body axis roll rate, rad/s
r_b = body axis yaw rate, rad/s
ϕ = body axis roll angle, rad
δ_a = aileron angle, rad
δ_r = rudder, rad
r_s = stability axis yaw rate, rad/s

$\left(\frac{A \mid B}{C \mid D}\right) =$

−0.0868	0.215	−0.977	0.0539	0	0.0179
−32.3	−0.374	2.40	0	6.35	6.66
1.06	−0.0406	−0.0809	0	1.71	−1.18
0	1	0.220	0	0	0
0	1	0	0	0	0
0	0	0	1	0	0
0	−0.215	0.977	0	0	0

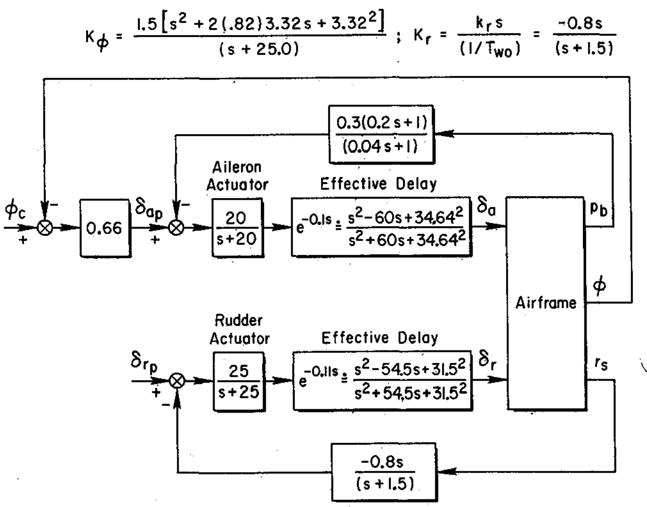


Fig. 1 Example lateral FCS.

Robustness of Stability with Respect to Input Uncertainties

An uncertainty model used for the purpose of preliminary flight control design approximates the unmodeled high-frequency dynamics of filters, sensors, structural modes, notch filters, and so forth, with effective time delays. These same effective delays model sampling delays and actuator rate limits. The low-frequency approximations will be adequate if these high-frequency dynamics are well above the magnitude crossover frequencies that define the bandwidths of the FCS feedback loops. Thus, in Fig. 1, the rudder and roll control channels each contain a nominal effective time delay (0.11 and 0.10 s, respectively).

From a realistic point of view there are many uncertainties and variations that can occur at many points around the FCS loops. Stability derivatives should be included in the above list, as well as critical parameters in the control system. A complete uncertainty model would account for each source in its appropriate location. Here, for concreteness, we consider only the case of effective time delays at the inputs to the roll control and rudder.

The multivariable uncertainty model is

$$y(s) = G(s)[I + E_m(s)]u(s) \tag{1}$$

where

$$I + E_m(s) = \text{diag}\{e^{-s\Delta\tau_a}, e^{-s\Delta\tau_r}\} \tag{2}$$

$$\bar{\sigma}[E_m(s)] = |e^{-s\Delta\tau} - 1|, \qquad \Delta\tau = \max(\Delta\tau_a, \Delta\tau_r) \tag{3}$$

Lump the separate controllers together using the multivariable notation:

$$u(s) = K(s)[r(s) - y(s)] \tag{4}$$

It then follows[1,2] that the closed-loop system is robustly stable with respect to $E_m(s)$ if

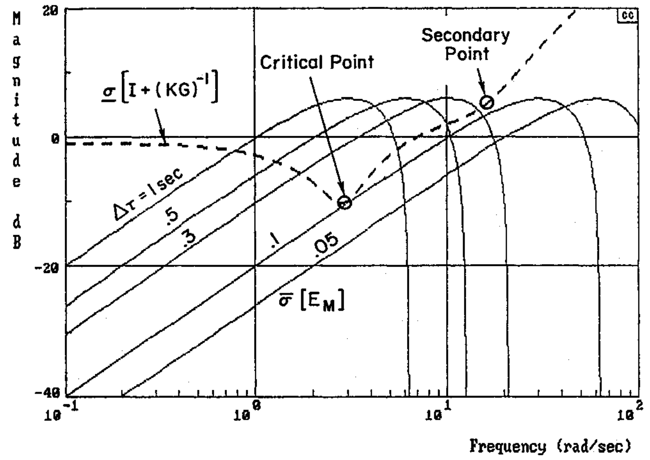
$$\bar{\sigma}[E_m(j\omega)] < \underline{\sigma}\{I + [KG(j\omega)]^{-1}\} \quad \text{for all } \omega \tag{5}$$

where $\bar{\sigma}$ and $\underline{\sigma}$ respectively denote the maximum and minimum singular values.

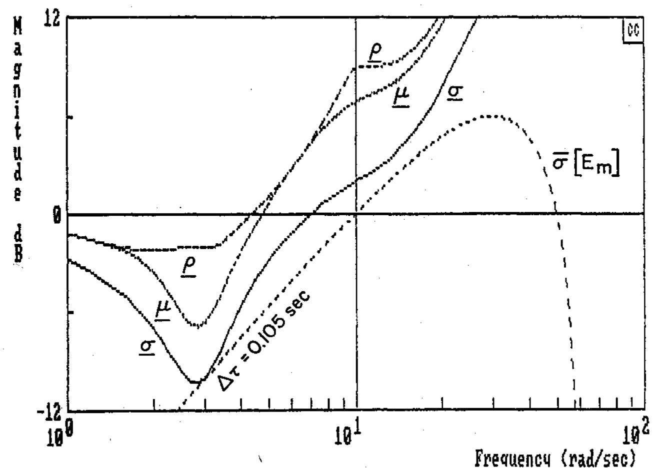
The final condition (5) is *not* conservative in the sense that a perturbation $E_m(s)$ is guaranteed to exist that just barely exceeds the frequency domain bound and results in closed-loop instability. The same condition, however, is conservative in the sense that a perturbation $E_m(s)$ with a particular structure can violate the frequency domain inequality without causing instability. Additional structure has been imposed in Eq. (2) on $E_m(s)$, namely that 1) the uncertainties in the actuator channels are delays and not some other type of dynamic element, and 2) there is no crosscoupling of the uncertainties, e.g., uncertainty in the roll actuator does not directly lead to uncertainty in the rudder position.

A graphical illustration of the robustness condition (5) is shown in Fig. 2a. The perturbation bounds $\bar{\sigma}(E_m)$ are shown for several $\Delta\tau$, and it is concluded that any combination of effective delays $\Delta\tau_a$ and $\Delta\tau_r$ such that both are less than about 0.1 s is not destabilizing. It is noted that in the critical frequency ranges

$$\bar{\sigma}[E_m(s)] \approx \Delta\tau s \tag{6}$$



a) Singular value test



b) Singular value $\underline{\sigma}$, spectral radius $\underline{\rho}$, and structured singular value $\underline{\mu}$

Fig. 2 Graphical robust stability tests.

Because of the particular diagonal structure used here it is expected that one or both of the effective delays can exceed this bound without destabilizing the system. Emerging robustness procedures[3,4] help to alleviate the conservativeness by taking advantage of additional structure on E_m. The singular value test (5) assumes no structure. If a diagonal structure is assumed for $E_m(s)$, where each diagonal element is the same, then a spectral radius test can be used:

$$E_m = (e^{-s\Delta\tau} - 1)I$$

$$\bar{\sigma}[E_m] < \underline{\rho}\{I + [KG(j\omega)]^{-1}\} \quad \text{for all } \omega \tag{7}$$

The spectral radius ρ is defined as the absolute value of the minimum eigenvalue, i.e., $\underline{\rho}[A] = |\underline{\lambda}[A]|$. (This definition is nonstandard, hence the underline, in order to keep the same basic form as the singular value test (5). More typically $\rho = |\bar{\lambda}[A]|$, in which case $\underline{\rho}[A] = 1/\rho[A^{-1}]$.)

If a block diagonal structure is assumed for E_m then the structured singular value is used. In Eq. (2), two 1×1 blocks are defined, which results in the following robustness test:

$$\bar{\sigma}[E_m] < \underline{\mu}\{I + [KG(j\omega)]^{-1}\} \quad \text{for all } \omega \tag{8}$$

A definition of the structured singular value $\underline{\mu}$ which suffices in this case is[3,4]

$$\underline{\mu}(A) = \underline{\sigma}(\min_e D^{-1}AD) \tag{9}$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$$

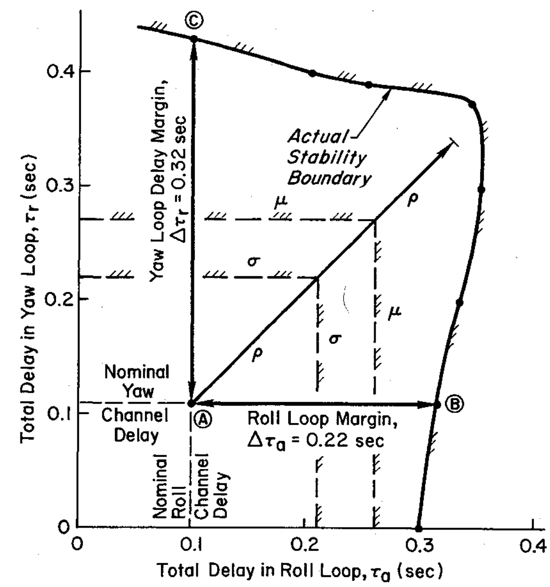


Fig. 3 Stability boundaries in roll-yaw time delay parameter plane.

The parameter e minimizes the Frobinius norm of $D^{-1}AD$, where the Frobinius norm is the square root of the element magnitudes squared. (This definition is again nonstandard, more typically $\mu[A] = \bar{\sigma}(\min_e D^{-1}AD)$ and $\underline{\mu}[A] = 1/\mu[A]$.)

The diagonal scaling e can be given the interpretation of scaling the input variable δ_r relative to δ_a. The singular value test is not invariant to this type of scaling, and the innovation of the structured singular value test is to minimize over all possible scalings. This definition and interpretation holds for the cases of 2 and 3 diagonal blocks, but not for more general block structures.

Graphical applications of the robustness tests [Eqs. (5), (7), and (8)] are shown in Fig. 2b. For clarity, only one uncertainty bound $\bar{\sigma}[E_m]$ is shown. The singular value test guarantees uncertainties $\Delta\tau$ up to 0.105 s in each channel. The spectral radius test improves this up to 0.24 s simultaneously in each channel, but nothing formal can be implied from this test for different delays in each channel. The structured singular test guarantees $\Delta\tau$ up to 0.15 s in each channel.

To visualize better the conservativeness issue, it is useful to compare the guaranteed delays from the tests in Eqs. (5), (7), and (8) with the actual stability boundaries shown in the rudder-aileron time delay parameter plane of Fig. 3. The stability boundary is determined by varying the delay (modeled as a second-order Pade approximation) in the rudder channel, closing the yaw damper loop, and then determining the delay margin in the roll loop using an open-loop Bode plot. Point A in Fig. 3 is the nominal condition, and points B and C represent conventional single axis delay margins. The conservativeness of the singular value test has been pointed out in many previous studies and is apparent from Fig. 3. Perhaps surprising is that in this example the structured singular value test is also conservative. This can be alleviated by using a real parameter singular value test,[11] which takes into account the fact that $\Delta\tau$ is a real parameter, but this is beyond the scope of the paper.

The preceding discussion has focused on how the additional structure of the perturbation can be used to decrease the conservativeness of the robustness tests. Questions like this have been raised and for the most part answered[3,4] by members of the theoretical community. We do not want to minimize the importance of these tests, indeed we want to increase their use, but the types of questions asked by researchers are not the ones that designers are most likely to ask. For starters, given the simple approximation for $\bar{\sigma}[E_m]$ in Eq. (6), what is a similar approximation for the other portions of the robustness tests of Eqs. (5), (7), and (8)? What are the low- and high-frequency approximations? What open-loop mode creates the critical point around 3 rad/s? What is the mode around 13 rad/s that is close to becoming critical? What are the critical aircraft and controller parameters that determine these critical points? Are the stability difficulties cited in the example responsible for the critical points? How sensitive are these to changes in the nominal flight control system, and how would matters be changed if the aerodynamics were modified? We now turn to an approach to explore these types of questions.

Motivation for Literal Formulations

The objective is to develop greater FCS-engineering insight into the new frequency domain robustness measures. The approach is to give literal (symbolic) expressions for the singular-value-based tests. This approach involves two concepts that have long been used for traditional multivariable flight control design—literal approximate factors and transfer function numerators of higher kinds, also known as coupling numerators.[9]

Over the years, literal approximate factors have been developed for a wide variety of conventional and VSTOL aircraft, rotorcraft, and other vehicles. These expressions give approximate relations for transfer function poles and zeros literally in terms of stability derivatives. This is best appreciated by an example such as the lateral-directional example being used here, where all of the following derivatives are defined in the stability axes[9]:

Dutch roll pole:

$$\omega_d \approx \sqrt{N'_\beta} \tag{10}$$

$$2(\zeta\omega)_d \approx -(Y_v + N'_r) - \frac{L'_\beta}{N'_\beta}\left(N'_p - \frac{g}{U_0}\right) \tag{11}$$

Roll-due-to-lateral controller dipole:

$$\frac{\omega_\phi}{\omega_d} \approx \sqrt{1 - \frac{N'_{\delta_a}}{L'_{\delta_a}} \cdot \frac{L'_\beta}{N'_\beta}} \tag{12}$$

$$2(\zeta\omega)_\phi \approx -(Y_v + N'_r) + \frac{N'_{\delta_a}}{L'_{\delta_a}} L'_r \tag{13}$$

While expressions such as these are often accurate to a few percent for numerical calculation, this is not the primary purpose for using literal approximate factors. The real value is in viewing the connections between the aircraft poles and zeros and the stability derivatives that are not available from a strictly numerical calculation of system eigenvalues. This knowledge, in turn, indicates: 1) the physical origins, nature, vehicle configuration dependence, and variation with flight condition/configuration of the vehicle poles and zeros; 2) the physical origins, nature, etc., of the limiting dynamical characteristics (closed-loop modes at high gains) corresponding to a particular choice of FCS feedback architecture; 3) possible control law components to adjust particular modes or effective numerators; and 4) the absolute and relative importance of uncertainties in particular derivatives.

Literal Approximations of Open-Loop Singular Values

With the concepts of literal approximate factors as background, we can now discuss the basic approach for developing literal approximations of singular values. The open-loop case is treated first, which is useful in its own right for performance analysis, and which is a starting point for the more complicated closed-loop case.

A simplifying change is first made to the aircraft example by assuming that ϕ and p_b are related by the ideal linearized kinematics $p_b = s\phi$ (rather than $p_b = s\phi - r_b \tan\gamma_0$). Compute p_b by differentiating the ϕ measurement, which reduces the number of feedback compensators to K_ϕ and K_r as noted in Fig. 1. This change is consistent with the use of an integrated sensor package. The open-loop combined aircraft controller system is:

$$KG = \begin{pmatrix} 0 & K_\phi(s) & 0 \\ 0 & 0 & K_r(s) \end{pmatrix} \frac{1}{\Delta(s)} \begin{pmatrix} N^{p_b}_{\delta_a}(s) & N^{p_b}_{\delta_r}(s) \\ N^{\phi}_{\delta_a}(s) & N^{\phi}_{\delta_r}(s) \\ N^{r}_{\delta_a}(s) & N^{r}_{\delta_r}(s) \end{pmatrix}$$

$$= \frac{1}{\Delta}\begin{pmatrix} K_\phi N^\phi_{\delta_a} & K_\phi N^\phi_{\delta_r} \\ K_r N^r_{\delta_a} & K_r N^r_{\delta_r} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

The maximum and minimun singular values of KG are given by

$$\bar{\sigma} = \sqrt{\bar{\lambda}}, \qquad \underline{\sigma} = \sqrt{\underline{\lambda}} \tag{15}$$

where $\bar{\lambda}$ and $\underline{\lambda}$ are the maximum and minimum eigenvalues of $[KG]^*[KG]$, and where * denotes the conjugate transpose. The eigenvalues for this type of matrix will always be real positive numbers. They are defined using the determinant identity

$$\begin{vmatrix} \lambda - |a|^2 - |c|^2 & -a^*b - c^*d \\ -ab^* - cd^* & \lambda - |b|^2 - |d|^2 \end{vmatrix}$$

$$= \lambda^2 - B\lambda + C = (\lambda - \bar{\lambda})(\lambda - \underline{\lambda}) \tag{16}$$

where

$$B = \bar{\lambda} + \underline{\lambda} = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

$$C = \bar{\lambda}\underline{\lambda} = |ad - bc|^2$$

If the eigenvalues are widely separated, say $\bar{\lambda} > 5\underline{\lambda}$, as is the usual case for lateral-directional examples such as this, then the following approximations for the minimum and maximum singular values are very good:

$$\bar{\sigma} = \sqrt{\bar{\lambda}} \approx \sqrt{B} \qquad \underline{\sigma} = \sqrt{\underline{\lambda}} \approx \sqrt{C/B} \tag{17}$$

In the rare cases when the eigenvalues are close together

$$\bar{\sigma} \approx \underline{\sigma} \approx \sqrt{\bar{\sigma}\underline{\sigma}} = \sqrt[4]{\bar{\lambda}\underline{\lambda}} = \sqrt[4]{C} \tag{18}$$

The geometric mean $\bar{\sigma}\underline{\sigma}$ can be exactly computed. In all cases the following inequalities are valid:

$$\bar{\sigma}_{\text{approx}} \geq \bar{\sigma} \geq \sqrt{\bar{\sigma}\underline{\sigma}} \geq \underline{\sigma} \geq \underline{\sigma}_{\text{approx}} \tag{19}$$

The geometric mean is included in this discussion for several reasons: It can be exactly determined, in some cases it approximates the minimum and maximum singular values, and it can be used as a definition of multivariable bandwidth.

Literal approximations of the open-loop singular values are obtained by substituting from Eq. (14) into Eq. (17) and by using the coupling numerator identity[9]

$$N^{\phi r}_{\delta_a \delta_r} = \frac{1}{\Delta}(N^\phi_{\delta_a} N^r_{\delta_r} - N^\phi_{\delta_r} N^r_{\delta_a}) \tag{20}$$

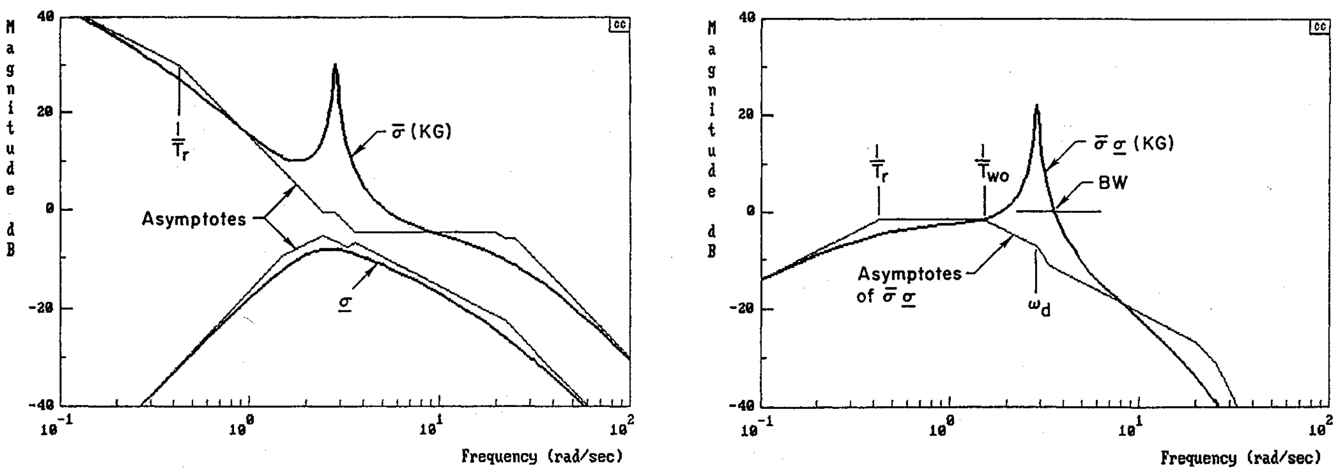
Resulting in:

$$[\bar{\sigma}(KG)]^2 \approx \left|\frac{1}{\Delta}\right|^2 \{|K_\phi|^2(|N^\phi_{\delta_a}|^2 + |N^\phi_{\delta_r}|^2) + |K_r|^2(|N^r_{\delta_r}|^2 + |N^r_{\delta_a}|^2)\} \tag{21}$$

$$[\underline{\sigma}(KG)]^2 \approx \frac{|K_\phi K_r N^{\phi r}_{\delta_a\delta_r}|^2}{|K_\phi|^2(|N^\phi_{\delta_a}|^2 + |N^\phi_{\delta_r}|^2) + |K_r|^2(|N^r_{\delta_r}|^2 + |N^r_{\delta_a}|^2)} \tag{22}$$

$$[\bar{\sigma}\underline{\sigma}(KG)]^2 = |K_\phi K_r N^{\phi r}_{\delta_a\delta_r}/\Delta|^2 \tag{23}$$

Square roots are needed as a final step. Square roots of transfer functions in general require noninteger powers of s, but not so here. The preceding equations are symmetric about the $j\omega$ axis and, therefore, the left-half-plane spectral factors can be used for the square roots.



a) Singular values

b) Geometric mean of singular values

$$\bar{\sigma}(KG) \approx \left| \frac{318\,(1.51)[.547,2.49][.947,3.64]\,(22.3)}{(.0048)(.427)(1.5)\,[.021,2.85]\,(20)(25)^2} \right| \quad \text{(approximate)}$$

$$\underline{\sigma}(KG) \approx \left| \frac{36.5\,(0)\,(.028)[.820,3.32]}{(1.51)[.547,2.49][.947,3.64]\,(22.3)} \right| \quad \text{(approximate)}$$

$$\bar{\sigma}\,\underline{\sigma} = \left| \frac{11600\,(0)\,(.028)[.820,3.32]}{(.0048)(.427)(1.5)[.021,2.85](20)(25)^2} \right| \quad \text{(exact)}$$

Fig. 4 Literal approximations for open-loop singular value.

Figures 4a and 4b contain Bode plots and transfer functions for the singular-value literal approximations. Exact numerical values for $\bar{\sigma}$ and $\underline{\sigma}$ in this airplane example are graphically indistinguishable from their literal approximations. As was the case discussed earlier for literal approximate factors, the importance of having literal approximations is *not* numerical; we would be just as happy with $\pm$ 10% accuracy. The importance is directly knowing how the singular values depend on the nominal aircraft dynamics. For example, the straightline asymptotes in Fig. 4 very clearly indicate the open-loop pole and zero locations. Several of the breaks are labeled, and it is seen that the dominant factors influencing the bandwidth are the yaw damper washout time constant T_{wo} and the Dutch roll damping ratio ζ_d. Individual poles and zeros are labeled below for $\bar{\sigma}\underline{\sigma}$. Poles and zeros corresponding to the effective delays do not appear because they cancel when the left-half-plane spectral factor is computed.

$$\bar{\sigma}\underline{\sigma} = \left| \frac{\begin{array}{ccc} & \text{Coupling zero} & \\ & 1/T_{\phi r} & \text{Lead in } K_\phi \\ 11600\ (0) & (0.0281) & [0.820,3.32] \end{array}}{\begin{array}{ccccc} (0.0048) & (0.427) & (1.5) & [0.021,2.85] & \underbrace{(20)(25)^2} \\ -1/T_s & 1/T_r & 1/T_{wo} & \zeta_d,\omega_d & \\ \text{Spiral} & \text{Roll} & \text{Washout} & \text{Dutch roll} & \text{Actuators} \end{array}} \right| \tag{24}$$

Further insight can be gained by concentrating on asymptotes and particular frequency ranges. Take, for example, the low-frequency asymptote of $\underline{\sigma}$. Use the following approximations from Ref. 9: 1) the washed out $K_r(s)$ is insignificant below $1/T_{wo}$ and 2) generally $|N^{\phi}_{\delta_r}| \ll |N^{\phi}_{\delta_a}|$, to obtain the following:

$$\underline{\sigma} \approx |K_r| \left| \frac{N^{\phi r}_{\delta_a \delta_r}}{N^{\phi}_{\delta_a}} \right| = \frac{k_r N'_{\delta_r}}{(N'_\beta / T_{wo})} s^2 \tag{25}$$

Hence, it can be seen that the washout time constant T_{wo} and the airframe dynamics N'_{δ_r} and N'_β all contribute to the slope of $\underline{\sigma}$ at low frequency.

The intent of the preceding discussion has been to show that much can be learned about singular values from literal approximations, and that much is to be gained by uniting classical and modern multivariable techniques for flight control system analysis and synthesis. For the example given, the actual singular values, which in general can only be obtained by assigning numerical values for all of the parameters of the system, are shown to be bounded and/or approximated by relatively simple, highly insightful, literal expressions.

Literal Formulation of the Robust Stability Criterion

To examine the criterion for robust stability as formulated in Eq. (5) we must take the additional step of developing literal expressions for the singular values of the inverse return difference. The derivation is similar to Eqs. (14–23), using $I + (KG)^{-1}$ instead of KG:

$$I + (KG)^{-1} = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \tag{26}$$

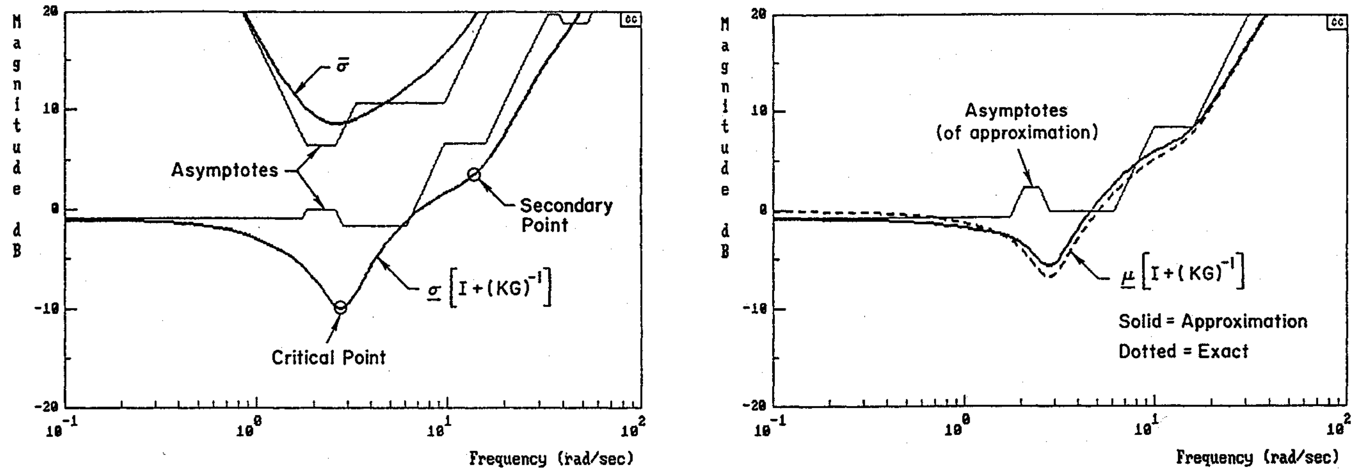
$$|\lambda I - [I + (KG)^{-1}]^*[I + (KG)^{-1}]| = \lambda^2 - B\lambda + C = (\lambda - \bar{\lambda})(\lambda - \underline{\lambda}) \tag{27}$$

The polynomial coefficients are

$$B = \bar{\lambda} - \underline{\lambda} = |a|^2 + |b|^2 + |c|^2 + |d|^2$$
$$= \left| 1 + \frac{1}{K_\phi} \frac{N^r_{\delta_r}}{N^{\phi r}_{\delta_a \delta_r}} \right|^2 + \left| \frac{1}{K_r} \frac{N^{\phi}_{\delta_r}}{N^{\phi r}_{\delta_a \delta_r}} \right|^2 + \left| \frac{1}{K_\phi} \frac{N^r_{\delta_a}}{N^{\phi r}_{\delta_a \delta_r}} \right|^2 + \left| 1 + \frac{1}{K_r} \frac{N^{\phi}_{\delta_a}}{N^{\phi r}_{\delta_a \delta_r}} \right|^2 \tag{28}$$

$$C = \bar{\lambda}\underline{\lambda} = |ad - bc|^2 = \frac{|1 + K_\phi N^{\phi}_{\delta_a}/\Delta + K_r N^r_{\delta_r}/\Delta + K_\phi K_r N^{\phi r}_{\delta_a \delta_r}/\Delta|^2}{|K_\phi K_r N^{\phi r}_{\delta_a \delta_r}/\Delta|^2} \tag{29}$$

The literal expressions for the maximum and minimum singular values follow as before, namely that $\bar{\sigma} \approx \sqrt{B}$ and $\underline{\sigma} \approx \sqrt{C/B}$.



a) Singular values b) Structured singular value

$$\underline{\sigma}[I+(KG)^{-1}] \approx \left| \frac{.0031[.66,1.7][.26,2.8][.94,6.0][.53,16][.85,40][.83,53]}{[.81,1.8][.67,2.6][.93,9.7][.87,33][.86,37]} \right|$$

$$\underline{\mu}[I+(KG)^{-1}] \approx \left| \frac{.0048[.66,1.7][.26,2.8][.94,6.0][.53,16][.85,40][.83,53]}{[.81,2.0][.50,2.4][.72,9.9][.86,34][.85,42]} \right|$$

Fig. 5 Literal approximations for singular and structured singular value robustness tests.

The numerator expression for C is quite interesting—it is precisely the coupling numerator expansion for the multiloop closed-loop characteristic polynomial.[9] (This is demonstrated here for 2 × 2 systems but is conjectured to hold in general.) Hence the closed-loop poles are zeros of the literal approximation of $\underline{\sigma}$. This provides a fundamental connection between classical and modern multiloop approaches and allows us to dissect the problem in insightful new ways.

Figure 5a contains a Bode plot of the literal approximations for $\bar{\sigma}$ and $\underline{\sigma}$ (only $\underline{\sigma}$ is of interest for robustness analysis, $\bar{\sigma}$ is included only to show that the separation is large). The approximations are graphically indistinguishable from exact calculations. The straightline asymptotes included on the plot clearly indicate that the robustness weaklink is the dipole [0.26,2.8]/[0.67,2.6]. Improvement can be gained by increasing the 0.26 damping of the closed-loop pole.

Further insight can be gained by examining term-by-term the numerator expression for C. It turns out for this example that all the terms are significant, which indicates that open-loop airplane modal characteristics (roll subsidence and Dutch roll), the system numerators, and the controllers all contribute to the robustness of the system. This contrasts with high-gain systems that are dominated by the controller and numerator characteristics.

Perhaps surprisingly, the controller actuators and the time delay characteristics are *not* the limiting robustness feature. If, however, the controllers are changed so that the critical low point in the robustness test around 3 rad/s is increased, then the secondary critical point seen in Fig. 2 around 13 rad/s would be the limiting factor. Because of its frequency, this secondary point is a very strong function of the actuators and other high-frequency dynamics.

One of the important features of literal approximations is the ability to connect the analysis with aircraft configuration characteristics. For example, the critical low point in the robustness test occurs around 3 rad/s, which is approximately the natural frequency of the Dutch roll mode. The feedbacks have changed the Dutch roll damping ζ_d, but not so much the natural frequency ω_d. The location depends primarily on the directional stability N'_β, and secondarily on the aileron induced yawing N'_{δ_a} and the effective dihedral L'_β. Having thus isolated the key aircraft parameters we can also predict first-order robustness trends with changes in flight conditions. For example

$$\omega_d \approx \sqrt{N'_\beta} \approx \sqrt{\rho U} \tag{30}$$

From these relations we can see that the approximate location of the $\underline{\sigma}$ dip is proportional to the square root of dynamic pressure (neglecting Mach and aeroelastic variations on the nondimensional derivatives). Another reason for isolating key aircraft parameters is that effort can be focused either on better identification of these parameters or on desensitizing feedbacks.

A Literal Approximation of Structured Singular Values

The same techniques detailed for singular values can be extended to the structured singular values. Repeating for clarity

$$\underline{\mu}(A) = \underline{\sigma}(\min_e D^{-1}AD) \tag{31}$$

where

$$A = I + (KG)^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$$

In this 2 × 2 case the frequency dependent scaling parameter e can be analytically determined: $e = \sqrt{c/b}$. It follows in short order that

$$\underline{\mu} \approx \sqrt{C/B} \tag{32}$$

where

$$B = |a|^2 + 2|bc| + |d|^2 \qquad C = |ad - bc|^2$$

The numerators of $\underline{\sigma}$ and $\underline{\mu}$ are the same, because C has not changed. The denominator of $\underline{\mu}$ is always smaller, because always $2|bc| < |b|^2 + |c|^2$, hence as expected $\underline{\mu} \geq \underline{\sigma}$. A transfer function approximation (with integer powers of s) is obtained by eliminating $2|bc|$, which is valid in the aircraft example because $2\,|bc| \ll |a|^2 + |d|^2$.

The resulting literal approximation for $\underline{\mu}$, straightline asymptotes, and transfer function approximation are shown in Fig. 5b. (If $2|bc|$ is included in the literal calculation, then straightline and transfer function approximations cannot be made, but the frequency response is graphically indistinguishable from the exact calculation.) The use of structured singular values means that crossfeeds are not allowed in the perturbation. It is concluded from Fig. 5b that if perturbation crossfeeds are not allowed, then the system crossfeed numerators $N^{\phi}_{\delta_r}$ and $N^{r}_{\delta_a}$ (present respectively in b and c) are of reduced importance in determining robustness. This reduced importance is seen as a shift of the *weak* dipole [0.26,2.8]/[0.67,2.6] to [0.26,2.8]/[0.50,2.4].

The literal analysis is concluded by examining the terms that make up B in Eqs. (28) and (32). Because of the washout in the yaw damper it is expected that the K_r terms a and c are insignificant compared to the K_ϕ terms b and d. A numerical test verifies this expectation. In the structured singular-value expression b is also insignificant, because as seen above $2|bc| \ll |a|^2 + |d|^2$, leaving only d as the important term:

$$
\begin{aligned}
B &= |a|^2 + |b|^2 + |c|^2 + |d|^2 \\
&\approx |b|^2 + |d|^2 && \text{use for } \underline{\sigma} \\
&\approx |d|^2 = |K_r N^{r}_{\delta_r}/\Delta|^2 && \text{use for } \underline{\mu} \qquad (33)
\end{aligned}
$$

Eliminating terms like this, while not true in general, often helps identify key parameters for a particular problem.

Summary and Conclusions

The viewpoint espoused here is that recent advances in frequency domain multivariable robustness techniques, when combined with classical multivariable flight control system design procedures, have much to offer as additional means for the assessment of flight control system designs. An example of a lateral-directional flight control system is used to illustrate how regular singular values and structured singular values are used to determine robustness measures. The example is then used to illustrate the development of various singular-value-based quantities expressed using literal terms that define the aircraft and controller characteristics. The insights and connections exposed in these formulations illuminate the governing and underlying features of the system, and give specific emphasis to key and critical robustness issues present in a particular design.

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